MATH-UA.251.001: Introduction to Mathematical Modeling
Homework 4

Discrete Dynamical Systems

The past week in class we have discussed a class of problems called **discrete dynamical systems**. Apart from finding explicit solutions to these type of equations, we would also like to study the qualitative behavior of discrete dynamical systems. We really would like to see how these systems evolve over a selected period of time. Recall in class that I said dynamical systems could exhibit a combination of four types of behavior: Evolution to a fixed point, Growth without bound (or "blow up"), Oscillation or Chaos. We want to investigate first whether the system evolves to a fixed point, or **equilibrium value**, which is the simple case.

Consider the system $P_{n+1} = f(P_n)$. A discrete dynamical system attains an **equilibrium (fixed) point** when there is no change in variable from one iteration to the next. Equivalently,

$$P_{n+1} = P_n$$

or, if we note that $\Delta P_n = P_{n+1} - P_n$, when

$$\Delta P_n = 0$$

This occurs where there is a solution to the equation $P_e = f(P_e)$, where P_e is the equilibrium value of the system. Starting with $P_0 = P_e$ (with the initial condition as the equilibrium value), indicates that $P_n = P_e$ for all n.

Now consider the model for discrete Malthusian growth,

$$P_{n+1} = (1 + r)P_n$$

where r is the growth rate of the system. We already know that it has an explicit solution of the form $P_n = (1+r)^n P_0$. What can using an equilibrium analysis tell us about the system without using or finding this solution? If we plug in P_e for P_{n+1} and P_n, we get

$$P_e = (1 + r)P_e$$

which simplifies to

$$rP_e = 0$$

We know the growth rate r cannot be zero, so this indicates that $P_e = 0$. What does this mean qualitatively? This means that the only equilibrium value is the **trivial case**: when you start a population with zero members, it stays at zero. This system has no unique equilibrium values, which means that it does not evolve to a steady state, and so must exhibit one of the other types of behavior. In this case that behavior is growth without bound. We'll look at how to analyze all cases over the next few weeks.

Stability Analysis

We would also like to know how to analyze the **stability** of the equilibria once we find them. This begins with an investigation of the case when instead of starting exactly at the equilibrium value, we start close to the equilibrium value. A system is considered **stable** if when you introduce a small change to the initial condition, $P_0 + \epsilon$, the system exhibits the same behavior as it would with the original initial condition, i.e. it will converge to the same fixed point. So if we start near the equilibrium, we will stay near the equilibrium. If the equilibrium is **unstable**, solutions will move away from the equilibrium. We consider the small change ϵ to be a **perturbation** to the system. More rigorously, recall $P_{n+1} = f(P_n)$. If we replace P_n with P and take the derivative with respect to P of $f(P)$, or $\frac{df(P)}{dP}$, we can qualitatively analyze the general flow of the solution. By example, consider the model for Malthusian growth

$$P_{n+1} = (1+r)P_n$$

Replacing P_n with P we get

$$f(P) = (1+r)P$$

Taking the derivative yields

$$\frac{df(P)}{dP} = 1 + r$$

We want to evaluate the flow of the system around our equilibria, so we then plug in $P = P_e$. Since the original equation here is linear, the derivative yields no dependence on P, and so we have nothing to plug into, but this is not always the case. We have the following cases, which come directly from our knowledge of convergence of sequences:

- If $f'(P_e) > 1$, the solutions to the discrete dynamical system grow away from the equilibrium without bound monotonically. This indicates the equilibrium is **unstable**.
- If $0 < f'(P_e) < 1$, the solutions to the discrete dynamical system approach the equilibrium. This indicates the equilibrium is **stable**.
- If $-1 < f'(P_e) < 0$, the solutions to the discrete dynamical system oscillate about the equilibrium while approaching it. This indicates the equilibrium is **stable**.
- If $f'(P_e) < -1$, the solutions to the discrete dynamical system oscillate, but move away from the equilibrium. This indicates the equilibrium is **unstable**.

Thus, for the Malthusian growth model, $1 + r > 1$ implies monotonic growth away from the equilibrium and instability, $0 < 1+r < 1$ implies the solutions approach the equilibrium and stability, etc. This is an excellent way to see how a complex system evolves in time without actually finding explicit solutions.

1. We will first build a model and then analyze it. Consider a population of rats in particular area of Manhattan - their population at time t will be represented as R_t. We will first consider a model with fixed migration and predation. We assume a fixed amount of rats, α, leave the population at generation t (whether via death or actual spatial movement) and a fixed amount of rats, β, add to the population at generation t (via migration - we will take care of birth rate with Malthusian growth). If we assume that the change in population is proportional to a growth rate (birth rate in this case), r, plus the migration factors, we have

   $$\Delta R_t = R_{t+1} - R_t = rR_t - \alpha + \beta$$

   We can write this as

   $$R_{t+1} = (1 + r)R_t - \alpha + \beta$$

which is in the form of a **discrete linear difference equation**. Give the solutions R_2, R_3 and R_4 in terms of the initial population, R_0 and the parameters α, β and r. Extend this result to the general solution, R_t in terms of R_0 and the parameters α, β and r.

2. Find the equilibria, or fixed points, for this system. What is the physical significance for these equilibria?

3. How do the parameters r, α and β effect the stability of the equilibria?

4. Assume no new rats are added to the population via migration, when $\beta = 0$, and the α represents a fixed amount of rats that humans eradicate with every generation. If we want to control the rat population but not have them go to extinction, is having α fixed a viable strategy?

5. Now let's assume a functional relationship for predation - humans want to control the population of rats by removing a an amount every generation proportional to their previous generation size, so $\alpha = aR_t$. Still assume $\beta = 0$. How does this change the equilibria and the stability of the system? How about when $\beta \neq 0$?

6. How might you modify our function α to maintain a stable population size of 1000 rats?

7. Read Section 1.2, 1.3, 2.2, 2.3 of Nonlinear Dynamics and Chaos (you already read 2.1). Steve Strogatz gives very accessible descriptions of Dynamical Systems.

8. Read "Common Wisdom, Logarithmic Differentiation and Compound Interest ".

9. Read "The Ricker Salmon Model".

   BONUS: Implement a version of the Rat Dynamical System in MATLAB with your own choice of α, β and r. Justify your choices of parameters.