

Common Wisdom, Logarithmic Differentiation, and Compound Interest

Peter A. Lindstrom

Math-Tech Division
North Lake College
Irving, TX 75038

Richard G. Montgomery

Mathematics Dept.
Southern Oregon State College
Ashland, OR 97520

Recently Michael Reed criticized the calculus curriculum for teaching how to differentiate functions of the form $f(x)^{g(x)}$ without providing any real applications [Reed 1987]. Without endorsing Reed's sentiment, we present an everyday application of such a differentiation.

The return on money invested at a given annual rate of interest varies with the frequency of compounding. Common wisdom says: The more frequent the compounding, the greater the return; and "continuous compounding" beats all. True, but texts do not prove these facts. They may carefully

- derive the familiar formula $P(1+r/n)^n$ for the year-end account balance after n compoundings of principal P at annual interest rate $r/(= 100r\%)$,
- resort to a table of computations for a few selected values of n (such as $n = 1, 2, 4, 365, 525600$), and
- conclude with a hand-wave that common wisdom is correct.

This shaky five-example hand-waving can be smoothed out nicely with a short exercise in logarithmic differentiation.

For positive numbers P and r , let

$$A(x) = P \left(1 + \frac{r}{x} \right)^x, \quad \text{for } x > 0.$$

This differentiable function agrees with $P(1+r/n)^n$ for natural numbers n and (as shown below) has a positive derivative on $(0, \infty)$.

Hence, the sequence $A(n) = P(1 + r/n)^n$ is indeed increasing; common wisdom is verified.

To see that A' is positive on $(0, \infty)$, use logarithmic differentiation to obtain

$$A'(x) = A(x) \left[\ln \left(1 + \frac{r}{x} \right) - \frac{r}{x+r} \right], \quad \text{for } x > 0.$$

The factor $A(x)$ is positive. Name the second factor $f(x)$. Since

$$f'(x) = \frac{-r^2}{x(x+r)^2}$$

is negative for $x > 0$, f is decreasing on $(0, \infty)$. Moreover, $\lim_{x \rightarrow \infty} f(x) = 0$. Thus, f must be positive on $(0, \infty)$. And so $A' = Af$, the product of two positive factors, must also be positive on $(0, \infty)$.

Now that we have established that earnings are greater if the interest is compounded more times per year, we need a standard to tell just how good a deal we are getting. This is the role of the popularly advertised "effective annual rate of interest," the annual once-compounded rate needed to obtain the same return as the multiply-compounded investment.

For example, if P dollars is invested at 8% compounded semi-annually (i.e., twice), its value after one year is $P(1 + .08/2)^2 = P(1.0816)$ dollars (exactly). To produce the same amount of money in one year with the same investment of P dollars compounded once at annual rate $R/(= 100R\%)$ requires that

$$P \left(1 + \frac{R}{1} \right)^1 = P(1.0816),$$

so that $R = 0.0816 = 8.16\%$. We say that the effective rate of interest of 8% compounded semiannually is 8.16%. More precisely, for annual interest rate r and n compoundings per year, the *effective rate of interest* R is defined by

$$R = \left(1 + \frac{r}{n} \right)^n - 1.$$

Observe that for any principal P , the effective rate R is the solution to the equation

$$P \left(1 + \frac{R}{1} \right)^1 = P \left(1 + \frac{r}{n} \right)^n.$$

To verify that the effective rate increases with the number of compoundings, fix r , think of R as a function of n

$$R(n) = \left(1 + \frac{r}{n} \right)^n - 1,$$

and observe that $R(n) = A(n)/P - 1$, where $A(n) = P(1 + r/n)^n$. Since we have already proved that $A(n)$ increases with n , it follows that $R(n)$ does likewise.

Even though the year-end return on an investment increases with the number of compoundings during the year, Table 1 suggests, as has been known for several centuries, that the luxury of specifying an enormously large n will not create gaudy wealth. Jacob Bernoulli (ca. 1700) proposed the question: Through contracts specifying ever larger n , does the depositor gain unlimited return? [Toeplitz 1963, 23-28]. Today we simply observe that

$$\lim_{n \rightarrow \infty} A(n) = \lim_{n \rightarrow \infty} P \left(1 + \frac{r}{n} \right)^n = \lim_{m \rightarrow \infty} P \left[\left(1 + \frac{1}{m} \right)^m \right]^r,$$

where $m = n/r$,

$$= Pe^r.$$

A bank that advertises "continuous compounding" uses this theoretical limit to determine the year-end value of the account. Since $A(n)$ is an increasing sequence with limit Pe^r , continuous compounding does indeed return more than any finite number of compoundings; more of the common wisdom verified!

Moreover, the *effective rate of interest for continuous compounding* is defined to be the once-compounded rate 100R% needed to achieve the continuously compounded return Pe^r . Solving $P(1 + R/1)^1 = Pe^r$, we have $R = e^r - 1$. Since $e^r - 1$ is also the limit of the increasing sequence $R(n)$ of effective rates of interest, it follows without surprise that the effective rate of interest of continuous compounding is greater than that of any finite number of compoundings.

Table 1.
Year-end value A and the effective rate of interest R
for \$1 compounded n times, for interest rates of 10% and 100%.

n	$r = 10\%$		$r = 100\%$	
	A	R	A	R
1 (annually)	1.10000	10.000%	2.000	100.0%
2 (semiannually)	1.10250	10.250%	2.250	125.0%
4 (quarterly)	1.10381	10.381%	2.441	144.1%
365 (daily)	1.10516	10.516%	2.714	171.4%
continuously	1.10517	10.517%	$e \approx 2.718$	171.8%

